A CONCISE HISTORY OF THE THEORIES OF TIDES, PRECESSION-NUTATION AND POLAR MOTION (FROM ANTIQUITY TO 1950)

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"I do not know what I may appear to the world; but to myself I seem only to have been like a boy, playing on the seashore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me."

Isaac Newton (1727)

1. History of Tidal Theory

1.1. The First Tidal Discoveries

About 330 B.C. the Greek astronomer and explorer *Pytheas* made a long voyage, sailing from the western part of the Mediterranean Sea (where he lived in a Greek colony) to the British Isles. Observing the great ocean tides there, he made a fundamental discovery: The tides were in some way controlled by the Moon. This discovery can be considered the starting point of tidal research; it was published in Pytheas' "On the Ocean", now lost but quoted by other antique authors. Pytheas discovered not only that there were two high tides per lunar day, but also that the amplitude depended on the phases of the Moon.

The Greek scientists could not observe the tides at home because of their insignificance there. Nevertheless, around 150 B.C. the astronomer *Seleukos* found out that the two tides per day had unequal amplitudes when the Moon was far from the equator; this is what we today call the diurnal inequality. Seleukos was able to detect this phenomenon because his observations were made at the Red Sea, this being, according to modern tidal analyses, one of the few ocean areas where the diurnal inequality is relatively pronounced.

The Greek scientist *Poseidonios* (c. 135–51 B.C.) devoted a part of one of his written works to a review of the tidal knowledge of his time, including some of his own studies made at the Atlantic coast of Spain around 100 B.C. Poseidonios' work is no longer preserved but it is quoted by the Greek geographer *Strabon* (63 B.C.–c. 25 A.D.) in his impressive book "Geographika". This book appeared in the year 23 A.D. and here we can read the oldest still existing text on tidal theory, describing all the above-mentioned phenomena:

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When the moon rises above the horizon to the extent of a zodiacal sign [30°], the sea begins to swell, and perceptibly invades the land until the moon is in the meridian; but when the heavenly body has begun to decline, the sea retreats again, little by little; then invades the land again until the moon reaches the meridian below the earth; then retreats until the moon, moving round towards her risings, is a sign distant from the horizon.... The flux and reflux become greatest about the time of the conjunction [new moon], and then diminish until the half-moon; and, again, they increase until the full moon and diminish again until the waning half-moon.

If the moon is in the equinoctial signs [zero declination], the behaviour of the tides is regular, but, in the solstitial signs [maximum declination], irregular, in respect both to amount and to speed, while, in each of the other signs, the relation is in proportion to the nearness of the moon's approach.

Strabon further writes:

There is a spring at the [temple of] Heracleium at Gades [Cadiz], with a descent of only a few steps to the water (which is good to drink), and the spring behaves inversely to the flux and reflux of the sea, since it fails at the time of the flood-tides and fills up at the time of the ebb-tides.

This passage on reversed tides in a well is a remarkable one since it represents, in fact, the first observations of earth tides (in the form of tidal strain, as we will see later on). Although the phenomenon in the well had been known for a long time it appears that Poseidonios was the first scientist to study it, during his above-mentioned scientific travel to Spain. Poseidonios, while admitting that "the ebb-tide often occurs at the particular time of the well's fullness", did not believe that it really had anything to do with the tides. Strabon, however, discussing the problem in detail, arrives at the conclusion that the phenomenon somehow must be a tidal one.

More than two thousand years ago the most important characteristics of the tides were known, mainly due to Greek observations at the British Isles and in the Red Sea. But in what way were the tides created? It was to take 1600 years before the origin of this strange phenomenon began to be understood. Meanwhile, several unsuccessful attempts were made to explain it.

1.2. Search for the origin of tides: On the wrong track

The considerable tides around Britain interested *Bede the Venerable* (673–735), a learned English monk. In the beginning of the 8th century Bede discovered the phase lag of the ocean tides, realizing that each port had its own tidal phase. As to the origin of the tides he was of the opinion that the tide ebbed through the Moon blowing on the water, and flowed again when the Moon had moved a bit.

The first scientific attempt to explain the tides was made in the middle of the 13th century by the Arabian scientist *Zakariya al-Qazwini* (c. 1203–1283). In a book on the wonders of Creation he claims that the flowing tide is caused by the Sun and the Moon heating the waters, thereby making them expand. He describes this in the following way in the case of the Sun:

As to the rising of the waters, it is supposed that when the Sun acts on them it rarefies them, and they expand and seek a space ampler than that wherein they were before, and the one part repels the other in the five directions eastwards, westwards, southwards, northwards and upwards.

His hypothesis, however, failed to explain why the Moon, not the Sun, played the leading role.

The great difficulty in understanding how the Moon and the Sun could have an influence on the Earth made people look for completely new explanations of the tides. The whole idea of Moon and Sun acting in some way was distrusted. According to one idea the tides were caused by the great whirlpool, Malströmmen (the Maelstrom), off the coast of northern Norway: Low tide was just the consequence of the sea water disappearing into the whirl, while high tide occurred when the water reappeared from the whirl. Today we know that there is, in fact, a connection between the whirlpool and the tide – but it is the tide that causes the whirl, not the other way around!

After the (re)discovery of America it was suggested in 1557 by *Julius Caesar Scaliger* (1484–1558), an Italian scientist, that the tides were caused not only by the Moon, but also by the sea water oscillating between the coasts of America and Europe. The background to this suggestion might have been the resonance phenomena that by then were known to occur in some of the large lakes in Switzerland.

Johannes Kepler (1571–1630), the German astronomer, was convinced that the tides in some way depended on the Moon and the Sun. He claimed in 1609 in his "Astronomia nova" that the explanation was an attractive force of the Moon and the Sun, a force which he believed to be some kind of magnetism. Clearly, he was inspired by Gilbert's recent discovery of the magnetic field of the Earth.

However, *Galileo Galilei* (1564–1642), the Italian physicist and astronomer, was surprised that the great Kepler "became interested in the action of the Moon on the water, and in other occult phenomena, and similar childishness". Galilei himself believed, defending the Copernican theory of a rotating Earth in 1616 and 1632, that the tides were produced by the combined effect of the Earth's rotation around its axis and its orbital motion around the Sun. These motions would set the water on the Earth into oscillations, observed as tides:

> It must happen that in coupling the diurnal motion with the annual, there results an absolute motion of the parts of the surface which is at one time very much accelerated and at another retarded by the same amount.... Now if it is true (as is indeed proved by experience) that the acceleration and retardation of motion of a vessel makes the contained water run back and forth along its length, and rise and fall at its extremities, then who will make any trouble about granting that such an effect may – or rather, must – take place in the ocean waters?"

The French mathematician *René Descartes* (1596–1650) – often known by his Latin name *Renatus Cartesius* – supported a lunar origin of the tides, presenting in 1644 his own idea on how it all worked: The Moon and the Earth were each surrounded by a large vortex. The pressure exerted by the vortex of the Moon on that of the Earth was transmitted down to the Earth's surface, giving rise to the tides. However, the theory of vortices erroneously predicted a low tide when there was in reality a high tide, although it must be admitted that the picture was quite complicated because of the phase lag of the ocean tides.

An extended version of Galilei's theory was given in 1666 by *John Wallis* (1616–1703), an English mathematician. He suggested the tidal oscillations resulted from the Earth's rotation combined, not only with the Earth's motion around the Sun, but also with its motion around the centre of gravity of the Earth-Moon system. Thereby Wallis tried to include the influence of the Moon into the theory.

The whole thing was very confusing: If Moon and Sun did not control the tides, how did one explain all the observations? If the observations were correct, how did one explain that Moon and Sun could control the tides on the Earth?

1.3. AHA! GRAVITATION AND TIDES

The solution to the problem was given in 1687 when *Isaac Newton* (1642–1727), the English mathematician, physicist and astronomer, published the theory of gravitation in his "Philosophiae naturalis principia mathematica". The origin of the tides was the hitherto unknown attractive force of Moon and Sun (and all other celestial bodies) – gravitation. The tides were created by gravitation being different at different distances from the celestial body. Newton writes:

> But let the body S come to act upon it [the globe], and by its unequable attraction the water will receive this new motion. For there will be a stronger attraction upon that part of the water that is nearest to the body, and a weaker upon that part which is more remote.

With his theory Newton could explain the three fundamental properties of the tides: the main period of 12 lunar hours, the dependence of the amplitude on the lunar phases, and the diurnal inequality. To clarify the situation Newton constructed the figure which is shown in our Figure 1:

> For the whole sea is divided into two hemispherical floods, one in the hemisphere KHk on the north side, the other in the opposite hemisphere Khk, which we may therefore call the northern and the southern floods. These floods being always opposite the one to the other, come by turns to the meridians of all places, after an interval of 12 lunar hours. And seeing the northern countries partake more of the northern flood, and the southern countries more of the southern flood, thence arise tides, alternately greater and less in all places without the equator.

Newton also was able to calculate the tidal force of the Sun and the Moon, respectively. Newton considered this "force to move the Sea" in the sense of producing the full rise from low tide to high tide. For the Sun he found, by using the Sun's disturbing influence on the lunar orbit, that the tidal force was 1/12 900 000 of the force of gravity in case the Sun was at zenith or nadir, and at its mean distance. This is an excellent value. By analysing English tidal observations with respect to the ratio between spring tides and neap tides Newton found the tidal force of the Moon to be 4.5 times that of the Sun. The true value, however, is 2.2. Thus Newton overestimated the lunar tidal force by a factor of 2, approximately. This factor will pop up again in Section 2.3, where the luni-solar gravitation is discussed within the theory of precession.

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in Syzygiis Solſtitialibus quàm in Æquinoctialibus. In Quadraturis autem Solſtitialibus majores ciebunt æſtus quàm in Quadraturis Æquinoctialibus; eò quod Lunæ jam in æquatore conſtitutæ effectus maximè ſuperat effectum Solis. Incidunt igitur æſtus maximi in Syzygias & minimi in Quadraturas Luminarium, circa tempora Æquinoctii utriuſque. Et æſtum maximum in Syzygiis comitatur ſemper minimus in Quadraturis,ut experientiâ compertum eſt. Per minorem autem diſtantiam Solis à Terra, tempore hyberno quàm tempore æſtivo, fit ut æſtus maximi & minimi ſæpius præcedant Æquinoctium vernum quàm ſequantur, & ſæpius ſequantur autumnale quàm præcedant.

Pendent etiam effectus Luminarium ex locorum latitudine. Deſignet *ApEP* Tellurem aquis profundis undique coopertam; *C* centrum ejus; *P p*, polos; *AE* Æquatorem; *F* locum quemvis extra Æquatorem; *F f* parallelum loci; *D d* parallelum ei reſpondentem ex altera parte æquatoris; *L* locum quem Luna tribus ante horis occupabat; *H* locum Telluris ei perpendiculariter ſubjectum; *h* locum huic oppoſitum; *K*, *k* loca inde gradibus 90 diſtantia, *CH*, *Ch* Maris altitudines maximas menſuratas à centro Telluris; & *CK*, *Ck* altitudines minimas: & ſi axibus *Hh*, *Kk* deſcribatur Ellipſis, deinde Ellipſeos hujus revolutione circa axem majorem *Hh* deſcribatur Sphærois *HPKhpk*; deſignabit hæc figuram Maris quam proximè, & erunt *CF*, *Cf*, *CD*, *Cd* altitudines Maris in locis *F*, *f*, *D*, *d*. Quinetiam ſi in præfata Ellipſeos revolutione punctum quodvis *N* deſcribat circulum *NM*, ſecantem parallelos *Ff*, *Dd* in locis quibuſvis *R*, *T*, & æquatorem *AE* in *S*; erit *CN* altitudo Maris in locis omnibus *R*, *S*, *T*, ſitis in hoc circulo. Hinc in revo-

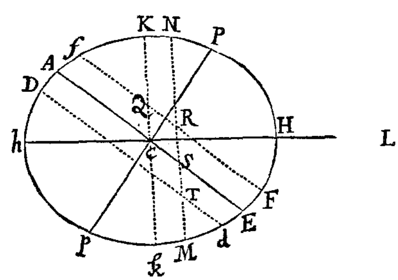

Fig. 1. A page from Newton's "Principia" (1687), with the figure to which the second quotation on the preceding page refers. The figure shows the tidally deformed ocean surface of the Earth: Pp denotes the axis of rotation, AE the equator, and L the direction to the Moon.

Newton's epoch-making discoveries meant that a foundation had been constructed for a mathematical treatment of tides. The Swiss mathematician and physicist *Daniel Bernoulli* (1700–1782) in 1740 wrote an essay on tides based on Newton's theory, but this was still not a great advance. However, Bernoulli was the one who found out that Newton had overestimated the ratio between the lunar and solar tides. Using French tidal observations he found a ratio of 2.5, close to the modern value.

The break-through for a mathematical theory of tides was made by *Pierre de Laplace* (1749–1827), the French mathematician and astronomer. He introduced the tidal potential, in a theory presented to the Royal Academy of Sciences in Paris 1775. Later on he extended this theory considerably and included it in his "Traité de mécanique céleste" of 1799. Here we find "Laplace's tidal formula", expressing the tidal potential as a function of latitude, declination and hour angle – see Figure 2. Let us listen to the description of Laplace:

> The three preceding terms give rise to three different types of oscillations. The periods of the oscillations of the first type are very long; they are independent of the rotational motion of the Earth, and depend only on the motion of the celestial body L in its orbit. The periods of the oscillations of the second type depend mainly on the rotational motion t of the Earth; they are one day approximately. Finally, the periods of the oscillations of the third type depend mainly on the angle 2t; they are about half a day.

Thus Laplace showed the tide to be mathematically separable into three different kinds of tides: long-periodical, diurnal and semi-diurnal. This separation has since then been a corner-stone in tidal theory.

Moreover, in the same work Laplace was the first to treat ocean tides as a problem of water in motion instead of water in equilibrium. His hydrodynamical equations, describing the propagation of tidal waves through the ocean, could not be solved in practice until the invention of the computer. Meanwhile, co-tidal charts were constructed using more or less unreliable methods.

1.4. Tidal friction and the length of the day

A completely new aspect of the tides – that of tidal friction – was introduced by the German scientist and philosopher *Immanuel Kant* (1724–1804). In 1754 he wrote an article in the Königsberg weekly magazine called "Ob die Erde in ihrer Umdrehung um die Achse einige Veränderung erlitten habe". Kant here realizes that the friction caused by the tidal motion of the ocean relative to the earth might cause a marked retardation of the Earth's rotation. He finds that this will go on until the Earth always turns the same side towards the Moon, i.e. until the length of the day is equal to a month. Kant writes:

> One can no longer doubt that the everlasting motion of the ocean from evening towards morning [from east towards west], a real and considerable force, will also always contribute something to decreasing the rotation of the Earth around its axis. This effect must inevitably become noticeable after a long period of time.
>
> As the Earth gradually approaches the standstill of its rotation, the period of this change will come to an end when the Earth's surface comes to a rest in relation to the Moon, i.e. when the Earth turns around its axis in the same time as that in which the Moon moves around the Earth.

Kant admits that he cannot present any evidence to support his hypothesis but leaves this as a task for others.

Although Kant claimed that it would be "a most shameful prejudice" not to

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La partie de $\alpha V'$ dépendante de l'action de l'astre L, est par le n°. 4, égale à

$$\frac{L}{4r^3}.\{\sin.^2 v - \tfrac{1}{2}.\cos.^2 v\}.(1+3.\cos.2\theta)$$
$$+\frac{3L}{r^3}.\sin.v.\cos.v.\sin.\theta.\cos.\theta.\cos.(nt+\varpi-\psi)$$
$$+\frac{3L}{4r^3}.\cos.^2 v.\sin.^2\theta.\cos.(2nt+2\varpi-2\psi).$$

Supposons que la partie correspondante de αy, soit égale à cette quantité multipliée par une indéterminée Q; ce produit étant de la forme $Y^{(2)}$, ou satisfaisant pour $Y^{(2)}$, à l'équation aux différences partielles,

$$0=\left\{\frac{d.\left\{(1-\mu\mu).\left(\frac{dY^{(2)}}{d\mu}\right)\right\}}{d\mu}\right\}+\frac{\left(\frac{ddY^{(2)}}{d\varpi^2}\right)}{1-\mu\mu}+6.Y^{(2)};$$

la partie de $\alpha V'$ correspondante à l'action de la couche fluide dont le rayon intérieur étant l'unité, le rayon extérieur est $1+\alpha y$, sera par le n°. 2, $\frac{4\pi}{5}.Y^{(2)}$, ou $\frac{3}{5\rho}.g.Y^{(2)}$; l'équation $\alpha gy=\alpha V'$, donnera donc,

$$\alpha y=\frac{L}{4r^3.g.\left(1-\frac{3}{5\rho}\right)}.\{\sin.^2 v-\tfrac{1}{2}.\cos.^2 v\}.(1+3.\cos.2\theta)$$
$$+\frac{3L}{r^3.g.\left(1-\frac{3}{5\rho}\right)}.\sin.v.\cos.v.\sin.\theta.\cos.\theta.\cos.(nt+\varpi-\psi)$$
$$+\frac{3L}{4r^3.g.\left(1-\frac{3}{5\rho}\right)}.\cos.^2 v.\sin.^2\theta.\cos.(2nt+2\varpi-2\psi).$$

Fig. 2. Laplace's tidal formula as it appears in the "Mécanique céleste" (1799). His interpretation of it is quoted on the previous page.

Five hundred and forty-third Meeting.

December 13, 1864. — MONTHLY MEETING.

The PRESIDENT in the chair.

The Corresponding Secretary read letters relative to exchanges.

Mr. Ferrel read the following paper.

Note on the Influence of the Tides in causing an Apparent Secular Acceleration of the Moon's Mean Motion.

As the unit of time depends upon the time of the earth's rotation upon its axis, any slight secular change in the time of its rotation, must cause an apparent secular acceleration or retardation of the moon's mean motion. There are two circumstances which may affect the time of the earth's rotation, first, the effect of the attractions of the sun and moon upon the tidal wave retarded by friction, secondly, a gradual decrease of the earth's volume from a loss of heat.

Fig. 3. Ferrel's introduction to his idea on the effect of tidal friction (1864).

bother about tidal friction, almost no one did so until a hundred years later, 1853. Then *William Ferrel* (1817–1891), an American oceanographer and meteorologist, pointed out that tidal friction causing a lengthening of the day, our unit of time, would lead to an apparent acceleration in the motions of celestial bodies. He tried to calculate this effect for the motion of the Moon, assuming the semi-diurnal ocean tide to have a mean phase lag of 30° (2 h).

A small acceleration of the Moon – observed through the study of ancient records of solar eclipses – had been detected already by Halley (1693). It was known, however, to be caused by the disturbing gravitational forces of the Sun and the planets. How, then, did Ferrel explain why his effect was not observed? Here a popular view at that time came in handy. The Earth was thought to cool down and, thereby, to contract. This would make the Earth rotate faster. Ferrel assumed that the effects of tidal friction and the Earth's cooling happened to balance each other, so that nothing could be observed!

At the same time as Ferrel published his paper an error was discovered in the complicated computations of the gravitational perturbations of the Moon's motion. When correcting this error it was found that half of the observed acceleration of the Moon no longer could be accounted for. This made Ferrel claim, in 1864 (Figure 3), that the residual acceleration could be explained by tidal friction causing a lengthening of the day amounting to 1 sec in 300 000 years. This would require a phase lag of only 2° (8 min).

Ferrel's work was published in America and, probably because of that, it was overlooked in Europe at that time. In 1865 the French astronomer *Charles Delau-*

nay (1816–1872), while redeveloping the lunar perturbation theory, arrived at a similar conclusion as Ferrel, apparently without knowledge of his work; it was through Delaunay the problem of tidal friction became widely known in scientific circles.

Soon after, in 1866, the English astronomer and geodesist *George Airy* (1801–1892) commented upon Delaunay's paper. Airy found that tidal friction, in addition to lengthening the day, should cause a growing distance of the Moon from the Earth. But the difficulties in handling these problems were overwhelming; this is nicely illustrated by Airy in a simple example:

> Conceive, for instance (as a specimen of a large class), a tide-mill for grinding corn. The water, which has been allowed to rise with the rising tide, is not allowed to fall with the falling tide, but after a time is allowed to fall, thereby doing work, and producing heat in the meal formed by grinding the corn. I do not doubt that this heat is the representative of vis viva [kinetic energy], lost somewhere, but whether it is lost in the rotation of the Earth or in the revolution of the Moon, I am quite unable to say.

Since this time the phenomenon of tidal friction has been the subject of more or less continuous scientific discussion; we will encounter it again in Section 1.6.

1.5. The discovery of earth tides

During the second half of the 19th century a vivid debate was going on concerning the internal constitution of the Earth. Was it fluid or solid? The English physicist *William Thomson* (1824–1907) – later known as *Lord Kelvin* – developed a theory of the Earth as an elastic solid. It appeared under the title "On the Rigidity of the Earth" in the Philosophical Transactions of the Royal Society of London in 1863. From his calculations Thomson arrives at the following conclusion:

> Hence it is obvious that, unless the average substance of the earth is more rigid than steel, its figure must yield to the distorting forces of the moon and sun, not incomparably less than it would if it were fluid.

Thus Thomson proposes the existence of earth tides; cf. Figure 4. According to Thomson the earth tides could be discovered and measured by observations of long period ocean tides. His idea was that the earth tides would reduce the observed amplitude of the ocean tides. Of these only the long period ones could be calculated theoretically, the diurnal and semi-diurnal ones being too disturbed by resonance phenomena.

At about the same time, in 1868, Thomson introduced the powerful tool of harmonic analysis into tidal theory. This led him to invent (four years later) the first tide prediction machine; it could handle 10 tidal constituents.

It was *George Darwin* (1845–1912; a son of Charles Darwin) who applied Thomson's ideas. Darwin was one of Thomson's students, specialising in astronomy and geophysics. He analysed tidal observations from 14 ports in England, France and India, together comprising 33 years of observations. Using the lunar fortnightly and monthly tides he was able to find the ratio of the height of the

II. "On the Rigidity of the Earth." By Professor WILLIAM THOMSON, F.R.S. Received April 14, 1862.

(Abstract.)

The author proves that unless the solid substance of the earth be on the whole of extremely rigid material, more rigid for instance than steel, it must yield under the tide-generating influence of sun and moon to such an extent as to very sensibly diminish the actual phenomena of the tides, and of precession and nutation. Results of a mathematical theory of the deformation of elastic spheroids, to be communicated to the Royal Society on an early occasion, are used to illustrate this subject. For instance, it is shown that a homogeneous incompressible elastic spheroid of the same mass and volume as the earth, would, if of the same rigidity as glass, yield about $\frac{7}{9}$, or if of the same rigidity as steel, about $\frac{2}{5}$ of the extent that a perfectly fluid globe of the same density would yield to the lunar and solar tide-generating influence. The actual phenomena of tides (that is, the relative motions of a comparatively light liquid flowing over the outer surface of the solid substance of the earth), and the amounts of precession and nutation, would in the one case be only $\frac{2}{9}$, and in the other $\frac{3}{5}$ of the amounts which a perfectly rigid spheroid of the same dimensions, the same figure, the same homogeneous density, would exhibit in the same circumstances. The close

Fig. 4. The first part of the abstract of Thomson's theory (1863) claiming the existence of earth tides.

ocean tide on the elastic Earth to that on a rigid Earth, i.e. the number which we today denote γ. He obtained $\gamma = 0.68 \pm 0.11$. His value happens to agree very well with modern values, but the main point is that Darwin showed it to be significantly smaller than 1. Thereby the existence of earth tides was proved. Darwin's historical result was first published in 1882 in "A Numerical Estimate of the Rigidity of the Earth" in Nature (see Figure 5); the full account of the tidal analysis was given in the next year. There he concludes:

> These results really seem to present evidence of a tidal yielding of the earth's mass, showing that it has an effective rigidity about equal to that of steel.

1.6. TIDES AND DYNAMICAL PROPERTIES OF THE EARTH

The concept of earth tides opened up new prospects for the research on tidal friction. Considering the Earth to be a viscous fluid, *George Darwin* studied tidal friction in the Earth's interior, instead of in the oceans. Darwin found, as Airy had done in the ocean case, that tidal friction will not only retard the Earth's rotation, but also cause the Moon to recede from the Earth:

A NUMERICAL ESTIMATE OF THE RIGIDITY OF THE EARTH[1]

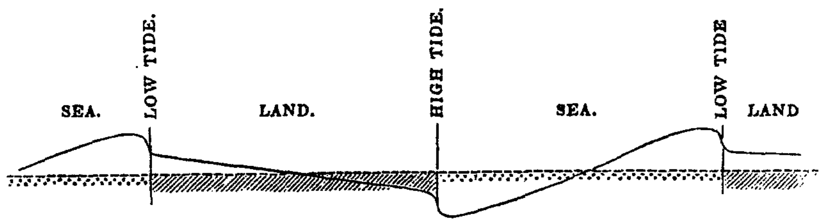
ABOUT fifteen years ago Sir William Thomson pointed out that, however it be constituted, the body of the earth must of necessity yield to the tidal forces due to the attraction of the sun and moon, and he discussed the rigidity of the earth on the hypothesis that it is an elastic body.

If the solid earth were to yield as much as a perfect fluid to these forces, the tides in an ocean on its surface would necessarily be evanescent, and if the yielding be of smaller amount, but still sensible, there must be a sensible reduction in the height of the oceanic tides.

Sir William Thomson appealed to the universal existence of oceanic tides of considerable height as a proof that the earth, as a whole, possesses a high degree of rigidity, and maintained that the previously received geological hypothesis of a fluid interior was untenable. At the same time he suggested that careful observation would afford a means of arriving at a numerical estimate of the average modulus of the rigidity of the earth's mass as a whole. The semi-diurnal and diurnal tides present phenomena of such complexity, that it is quite beyond the power

[1] Paper read by G. H. Darwin, F.R.S., at the British Association Southampton meeting.

Fig. 5. The beginning of Darwin's announcement (1882) of the discovery of the earth tides.



The straight line is a section of the undisturbed level, the shaded part being land, and the dotted sea. The curve shows the distortion, when warped by high and low tide as indicated.

Fig. 6. Ocean loading tides as illustrated by Darwin (1882).

The moon-earth system is, from a dynamical point of view, continually losing energy from the internal tidal friction. One part of this energy turns into potential energy of the moon's position relatively to the earth, and the rest developes heat in the interior of the earth.

Darwin's computations, published in 1879 as the first of a long series of papers on the subject, showed that in the early days of the Earth's history the Moon must have been very much closer to the Earth than now. Thus tidal friction was found to play a fundamental role in the evolution of the the Earth-Moon system; it even was found to raise questions as to the very origin of the Earth-Moon system.

Three years later, in 1882, Darwin tried to calculate the effect of ocean loading on the elastic crust. This led him to predict the existence of loading tides, i.e. earth tides due to loading from ocean tides. Darwin's illustration of the loading tides is shown in Figure 6.

In order to calculate loading tides one would need global co-tidal charts of the ocean tides. The first realistic such map was constructed by the American oceanographer *Rollin Harris* (1863–1918), who discovered and explained, in 1904, the general amphidromic character of the ocean tides.

A vertical tidal displacement of the crust should be accompanied by a corresponding tidal tilt. Many years of search using horizontal pendulums had elapsed when finally *Oskar Hecker* (1864–1938), a German geodesist and seismologist, succeeded in observing this tilt of only a hundredth of a second of arc, at Potsdam in 1907. Hecker's instrumentally obtained value of γ agreed with Darwin's old result, obtained in a quite different way. Furthermore, Hecker found a difference between the values in the N–S direction and the E–W direction, a difference which he could ascribe to the ocean loading tides suggested by Darwin.

A powerful method of describing tidal deformations of the elastic Earth was given in 1909 by the English geophysicist *Augustus Love* (1863–1940), and included in his book "Some Problems of Geodynamics" in 1911. Here we find what we now know as "Love's numbers" h and k, h characterizing the height of the deformation caused by the tidal potential, k characterizing the additional potential caused by the deformation. Love established the relation

$$\gamma = 1 + k - h$$

Furthermore, he – as well as Larmor – derived a relation containing k only (see Section 3.3), between the polar motion period of the elastic Earth and that of a rigid Earth. From the combination of tidal observations and polar motion observations Love was able to calculate the values of h and k: $h = 0.6$, $k = 0.3$. The value of h indicated a maximum vertical tidal displacement of 0.5 m.

A little later, in 1914, one of Hecker's geodetic colleagues, *Wilhelm Schweydar* (1877–1959), was the first to observe earth tides with a gravimeter. For the enhancement of the gravity variations due to the elasticity of the Earth he obtained a factor of $\delta = 1.20$, only slightly larger than the modern value ($\delta \approx 1.16$). Schweydar derived the relation

$$\delta = 1 + h - \tfrac{3}{2}k$$

From then on the combination of tilt and gravity observations, i.e. γ and δ, were to be fundamental for the determination of Love numbers.

An important contribution to the foundations of tidal theory was the harmonic expansion of the tidal potential made by the English mathematician and oceanographer *Arthur Doodson* (1890–1968) in 1921. It consisted of no less than 386 components of different periods and amplitudes.

Tidal friction, that had started as a problem of ocean tides, had been turned by Darwin into a problem of earth tides. By 1920 it was turned back into an ocean tidal problem by *Harold Jeffreys* (1891–1989), the English astronomer and geophysicist. Jeffreys brought forward evidence that the secular retardation of the Earth's rotation was caused mainly by tidal friction in shallow seas, a theory which occupied a strong position for quite a long time. It was an extension of a theory published the year before by an English meteorologist, *Geoffrey Taylor* (1886–1975), whose idea, curiously enough, originated from a dynamical similarity between tidal friction on an ocean bottom and wind friction along a ground covered with grass.

A different tidal effect on the Earth's rotation was introduced by Jeffreys in 1928. He found that the long period earth tides, forming small periodical variations in the flattening of the Earth, must cause corresponding variations in the rotational velocity large enough to affect accurate time-keeping.

A problem that had been unsolved ever since the days of Poseidonios and Strabon was that of tides in wells. How did they arise, and why did they usually have a phase opposite to that of the tidal potential? The problem was solved in 1940 by the American geophysicist *Chaim Leib Pekeris* (1908–):

> It is clear that at the time of the moon's transit, when the earth tide is high, the region underneath the station is under tension and is dilated, while six hours later, when the earth-tidal displacement is downwards, it is under compression. Compression in the water-bearing region would squeeze the water into the well and would thus bring about a rise of the water level at a time the displacement due to the earth tide is downwards.

Pekeris calculated that the volume strain, or dilatation, should be of the order of 10^{-8}. This formed the beginning of tidal strain research.

Already in 1905 a theory for the elastic deformations of a non-homogeneous Earth had been developed by a German mathematician, *Gustav Herglotz* (1881–1953), in 1920 extended to a compressible non-homogeneous Earth by the American mathematician *Leander Miller Hoskins* (1860–1937). It resulted in a differential equation of the sixth order which was, at that time, beyond the powers of anybody to solve. However, in 1950 the Japanese geophysicist *Hitoshi Takeuchi* turned the problem into three differential equations of the second order and succeeded in solving them numerically. Thereby he obtained, for the first time, tidal Love numbers for a realistic Earth model as given by seismological data.

The year before, Jeffreys pointed out that precession-nutation is caused by the diurnal tidal forces. A resonance in the liquid core of the Earth had been found by him to noticeably affect the nutation (Section 2.6). Consequently, the liquid core should affect the diurnal earth tides, too, causing the corresponding Love numbers to deviate from those computed by Takeuchi. This initiated the great challenge of finding, by theory as well as observations, the response to tidal forces of an elastic Earth, partly covered by oceans, with a liquid core.

2. History of Precession-Nutation Theory

2.1. The Discovery of Precession

During many years the Greek astronomer *Hipparchos* (c. 180–c. 120 B.C.) made observations of stars on the island of Rhodos. From these observations he created the first extensive star catalogue, containing nearly 1000 stars with their coordinates. When comparing some of his coordinates with those determined a century and a half earlier, he found a systematic decrease of the ecliptic longitudes of the stars. Hipparchos correctly interpreted this as a continuous motion of the equinoxes along the ecliptic. Thus Hipparchos had made a remarkable discovery: the precession. This phenomenon meant that the equator was not stable but moved in such a way that the celestial pole traced out a circle around the ecliptic pole.

Hipparchos' discovery was published about 125 B.C. in "On the Displacement of the Solsticial and Equinoctial Points". In this work he also calculated the precessional constant, i.e. the value of the annual precession. For the star Spica he knew the coordinates from Timocharis' old observations in 294 and 283 B.C. and his own observations in (at least) 146 and 135 B.C. Hipparchos writes:

Spica, for example, was formerly 8°, in zodiacal longitude, in advance of the autumnal [equinoctial] point, but is now 6° in advance.

This makes a precession of 2° in 148 years, i.e. 49″/year. The value is very close to the modern one, 50″/year. However, Hipparchos was well aware of its uncertainty. From all his calculations he considered 1° in 100 years, or about 36″/year, to be a minimum value of the precessional constant.

Hipparchos' work is no longer preserved but it is quoted by the Greek mathematician and astronomer *Klaudios Ptolemaios* (c. 100–c. 175 A.D.) in his famous book "Almagest". This book was written about 150 A.D. under the Greek name "Mathematike syntaxis" (Mathematical Systematic Treatise), later "Megale syntaxis" (Great Systematic Treatise). When translated into Arabic only the first word was retained, now in the form "megiste" (greatest), to which was added the Arabic definite article "al", finally leading to today's "Almagest". Ptolemaios here confirms the existence of the precession. But he does not accept Hipparchos' view of the precession as a movement of the equinoxes, probably because this view is related to the Earth as a rotating body. Ptolemaios was convinced that there was no such thing as a rotation of the Earth. To him, therefore, the precession is a movement of a celestial sphere onto which the stars seem to be fixed:

The sphere of the fixed stars also performs a motion of its own in the opposite direction to the revolution of the Universe.

The sphere of the fixed stars has a movement towards the rear with respect to the solsticial and equinoctial points . . . . This motion takes place about the poles of the ecliptic.

Ptolemaios also determined the precessional constant. However, the value which Hipparchos had considered to be the smallest possible one, 1° in 100 years, is adopted by Ptolemaios as the most probable one. He refers to his own observations leading to this value, but these observations seem to have suffered from a systematic error. His star catalogue suffers from the same systematic error throughout.

The "Almagest" was in many respects a master-piece, serving as a text-book for 1400 years. Consequently, Ptolemaios' view of the character of the precession and his value of the precessional constant were to mislead and confuse scientists during the same long period of time.

2.2. Precession and the rise and fall of trepidation

In the 9th century an Arabian mathematician, *Thabit ibn Qurra* (836–901), studied the "Almagest", even making a revision of one of the Arabic translations. The values of the precessional constant obtained up till then, including Ptolemaios' erroneous one, led ibn Qurra to the conclusion that this constant was not constant at all. So he had to introduce a large variation in the precession, known as trepidation. To account for this he suggested an additional moving celestial sphere.

The idea that the precession might be variable was not new, but from now on the trepidation was accepted as a real phenomenon by most scientists. However, one who expressed strong doubts about its reality was, the Arabian astronomer *Mohammed al-Battani* (858–929), in connection with making a redetermination of the precessional constant and a new star catalogue.

The machinery of moving celestial spheres adopted to explain precession and trepidation was not questioned for many centuries – not until 1543. That year marks the (re)discovery of the Earth's rotation. It was one of the basic ideas put forward by *Nicolaus Copernicus* (1473–1543), the Polish astronomer and priest, in his "De revolutionibus orbium coelestium" (the title page of which ended with the words: "Buy it, read it, enjoy it"!). This had important consequences for the view on the precession. Copernicus realized that the precession is a movement of the Earth itself:

> From the time of Ptolemaios to ours there has been a precession of the equinoxes and solstices of about 21°.
>
> The equinoxes seem to arrive before their time – not that the sphere of the fixed stars is moved eastward, but rather that the equator is moved westward, as it is inclined obliquely to the plane of the ecliptic in proportion to the amount of deflexion of the axis of the terrestrial globe.

Thus the precession now became a phenomenon associated with the axis of rotation of the Earth. According to Copernicus' theory the Earth moves in such a way that the rotational axis describes a conical motion around the normal to the ecliptic, with a period of 25 800 years.

Copernicus still believed, however, that the precession was accompanied by a trepidation. Using the works of Ptolemaios and al-Battani he found a period for the trepidation of 1 700 years. This result can be seen to be produced mainly by the ancient systematic error of Ptolemaios.

Before we leave Copernicus we should mention that he discovered what he himself called "an additional surprise of nature": the decrease of the obliquity of the ecliptic. Also this effect he believed to show a kind of variation, which he suspected was closely related to the trepidation.

The end of the deep-rooted notion of trepidation came with the Danish astronomer *Tycho Brahe* (1546–1601). He had erected an impressive observatory on the small island of Ven. By using his extremely accurate observations made here, and by critically going through the ancient observations, he arrived in 1588 at the definite conclusion that the trepidation did not exist.

So, one was now left with a uniform precession of the Earth. The cause of the precession was, however, still hidden in the dark. No one even seems to have made an attempt to find it.

2.3. AHA! GRAVITATION, FLATTENING, AND PRECESSION-NUTATION

In 1687 *Isaac Newton* (1642–1727), the English mathematician, physicist and astronomer, published the "Philosophiae naturalis principia mathematica" containing, among other things, his fundamental law of gravitation. In Section 1.3 we saw how this discovery enabled Newton to find the origin of the tides and explain their main characteristics. An even more remarkable achievement was that Newton found the origin of the precession. The precession turned out to be caused by gravitational forces of the Moon and the Sun in combination with a deformation of the Earth. Newton also realized that the Earth should be flattened at the poles as a consequence of its rotation. Thus the Moon's and the Sun's gravitation acting on the inclined Earth's equatorial bulge created the precession. A series of complicated arguments was presented by Newton to prove this result, which he himself expresses with the words:

> Redundant matter in the aequatorial regions of a globe causes the nodes to go backwards.

It is interesting to note an immediate application of the precession theory. We let Newton speak again:

> And thence from the motion of the nodes is known the constitution of the globe. That is if . . . the motion (of the nodes) be in antecendentia, there is a redundance of the matter near the equator; but if in consequentia, a deficiency.

Since the precession of the equinoxes (or the nodes) was observed to be a motion backwards ("in antecendentia") the Earth must necessarily be flattened at the poles, not at the equator. Yet, as we know, the Earth's flattening was to be a matter of great controversy for half a century.

Newton also made a theoretical calculation of the precessional constant. The beginning of this is shown in Figure 7. Assuming a hydrostatic equilibrium of the Earth he had found the Earth's flattening to be 1/231. Using this value he computed the solar precession. He then calculated the lunar precession in an ingenious way. He had shown that the gravitational attractions of the Moon and the Sun produce the precession – but he had also shown that the same forces produce the tides. So he simply uses the result of his analysis of ocean tidal observations (see Section 1.3) to find also the lunar precession:

> The remaining motion will now be $9'' \, 7''' \, 20^{iv}$ which is the annual precession of the equinoxes, arising

[470]

Prop. XXXIX. Prob. XIX.

Invenire Præceſſionem Æquinoctiorum.

Motus mediocris horarius Nodorum Lunæ in Orbe circulari, ubi Nodi ſunt in Quadraturis, erat 16″. 35‴. 16ⁱᵛ. 36ᵛ. & hujus dimidium 8″. 17‴. 38ⁱᵛ. 18ᵛ. (ob rationes ſupra explicatas) eſt motus medius horarius Nodorum in tali Orbe; fitque anno toto ſidereo 20 *gr.* 11′. 46″. Quoniam igitur Nodi Lunæ in tali Orbe conficerent annuatim 20 *gr.* 11′. 46″. in antecedentia; & ſi plures eſſent Lunæ motus Nodorum cujuſque, per Corol. 16. Prop. LXVI. Lib. I. forent reciproce ut tempora periodica; & propterea ſi Luna ſpatio diei ſiderei juxta ſuperficiem Terræ revolveretur, motus annuus Nodorum foret ad 20 *gr.* 11′. 46″. ut dies ſidereus horarum 23. 56′. ad tempus periodicum Lunæ dierum 27. 7 hor. 43′; id eſt ut 1436 ad 39343. Et par eſt ratio Nodorum annuli Lunarum Terram ambientis; ſive Lunæ illæ ſe mutuò non contingant, ſive liqueſcant & in annulum continuum formentur, ſive denique annulus ille rigeſcat & inflexibilis reddatur.

Fingamus igitur quod annulus iſte quoad quantitatem materiæ æqualis ſit Terræ omni *PapAPepE*, quæ globo *PapE* ſuperior eſt; & quoniam globus iſte eſt ad Terram illam ſuperiorem ut *aC qu.* ad *AC qu.* — *aC qu.* id eſt (cum Terræ diameter minor *PC* vel *aC* ſit ad diametrum majorem *AC* ut 689 ad 692) ut 4143 ad 474721 ſeu 1000 ad 114584; ſi annulus iſte Terram ſecundum æquatorem cingeret, & uterque ſimul circa diametrum annuli revolveretur, motus annuli eſſet ad motum globi interioris (per hu-

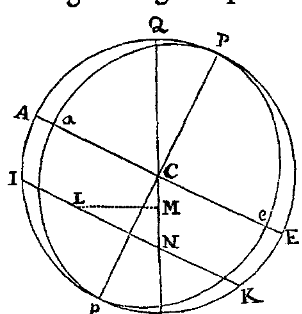


Fig. 7. The beginning of Newton's calculation of precession given in the "Principia" (1687). The figure shows the flattened Earth: Pp denotes the axis of rotation, or minor axis, AE the major axis, and QR the normal to the ecliptic.

from the force of the Sun. But the force of the Moon to move the sea was to the force of the Sun nearly as 4.4815 to 1. And the force of the Moon to move the equinoxes is to that of the Sun in the same proportion. Whence the annual precession of the equinoxes, proceeding from the force of the Moon, comes out 40″ 52‴ 52^{iv}, and the total annual precession, arising from the united forces of both, will be 50″ 00‴ 12^{iv}, the quantity of which motion agrees with the phaenomena.

Newton's results of 41″ and 9″ for the lunar and solar precession, respectively, agree pretty well with the actual figures, 34″ and 16″. Since Newton's ratio between the lunar and solar tides is about twice the real ratio the same thing applies to his lunar and solar precessions. Thus the excellent agreement between his total precessional constant and the observed one is somewhat illusory, being caused by different errors in numerical quantities cancelling each other.

Although Newton touched upon the existence of a small nutation, it was the English astronomer *James Bradley* (1692–1762) who discovered and explained the principal nutation of the Earth. Bradley had made repeated observations of the declinations of stars during 20 years. This observation series revealed a periodic variation of the declinations, the period being $18\frac{1}{2}$ years, the amplitude 9″, and the phase depending on the right ascension of the star. Bradley explained the variation by a nutation of the Earth, i.e. a movement of the Earth involving a periodic variation of the inclination of the Earth's axis. The nutation, he found out, was closely related to that part of the precession which was caused by the Moon. Bradley announced this discovery in 1748 in the Royal Society of London:

> I suspected, that the Moon's Action upon the Equatorial Parts of the Earth might produce these Effects: For, if the Precession of the Equinox be, according to Sir Isaac Newton's Principles, caused by the Actions of the Sun and Moon upon those Parts; the Plane of the Moon's Orbit being at one time above ten Degrees more inclined to the Plane of the Equator than at another; it was reasonable to conclude, that the Part of the whole annual Precession, which arises from her Action, would in different Years be varied in its Quantity.
>
> I perceived that something more, than a mere Change in the Quantity of the Precession, would be requisite to solve this Part of the Phaenomenon. Upon comparing my Observations of Stars near the Solsticial Colure, that were almost opposite to each other in Right Ascension, I found... that this apparent Motion, in both those Stars, might proceed from a Nutation in the Earth's Axis.

When informed by Bradley about his results, *John Machin* (c. 1680–1751), an English mathematician, suggested to him a useful geometrical method of describing the nutation. The true pole of the celestial equator moved in a little circle – the nutation circle – of radius 9″ with a period equal to that of the nodes of the Moon's orbit, 18.6 years, around a mean pole. This mean pole, in its turn, moved in the large precession circle of radius 23.5° with the period of 25 800 years around the pole of the ecliptic.

Bradley suspected that the nutation circle might be an ellipse instead, but considered that this problem could be solved only within the framework of a mathematical theory of precession-nutation. The time was now ripe for such theories.

2.4. The mathematicians enter the scene

Mathematical attacks on precession-nutation were made already the year after Bradley's discovery of the nutation. One was performed by the French mathematician *Jean le Rond d'Alembert* (1717–1783; as a new-born baby he was found abandoned at the church St. Jean le Rond, hence his name). In 1749 he presented his "Recherches sur la précession des équinoxes et sur la nutation de l'axe de la

Terre", which contains a detailed mathematical theory of precession and nutation. Here he makes use of the dynamical principle that he himself had found six years earlier. In particular d'Alembert shows that the nutation circle must be replaced by a nutation ellipse, the major axis of which is directed towards the pole of the ecliptic - cf. Figure 8. His computation of the semi-axes of the nutation ellipse yields 9″ and 6″, in close agreement with the modern values (9″ and 7″).

The other mathematical attack on precession-nutation made in the same year appeared in a paper with the same title as d'Alembert's book. The author this time was *Leonhard Euler* (1707–1783), the Swiss mathematician working in Russia and Germany. Euler arrived principally at the same results as d'Alembert. But in addition Euler found a new nutation term: one depending on the Sun, with a period of 1/2 year. This is the nutation already mentioned briefly by Newton which Euler was now able to calculate.

Nine years later, in 1758, Euler developed a general theory for rotating rigid bodies. Introducing the concepts of torque and moment of inertia Euler derived the equations of motion which today bear his name. This made it possible to deal with precession and nutation as a special solution of the Euler equations. Furthermore, an important consequence of these equations was the discovery of the phenomenon of polar motion; see further Section 3.1.

In 1799 *Pierre de Laplace* (1749–1827), the French mathematician and astronomer, published the first two volumes of his "Traité de mécanique céleste". Here he developed the mathematical theory of tides (Section 1.3). In the same work we find an extensive treatment of the theory of precession-nutation. Laplace applies Euler's equations mentioned above. In this way he derives precession-nutation formulae of the type illustrated in Figure 9. They show the central role played by the moment of inertia ratio $(C - A)/C$, reflecting the flattening of the earth ellipsoid. Furthermore, they show the possibility of expressing precession-nutation as a harmonic series, a possibility that could not be used very much until our own century.

2.5. Precession and the Ice Age

Around the middle of the 19th century a great interest in precession arose in an unexpected context: the Earth's climate. The background was the recent discovery of the Ice Age.

Searching for a cause of the Ice Age, *Joseph Adhémar* (1797–1862), a French mathematician, came up with the idea of the precession playing an essential role. He did so in a book called "Révolutions de la mer" which was published in 1842, five years after Agassiz' discovery of the Ice Age. Adhémar argues in the following way: Because of the eccentricity of the Earth's orbit around the Sun and the direction of the Earth's inclined rotational axis, the winters are longer in the southern hemisphere than in the northern one. The precession of the Earth – cf. Figure 10 – will cause this situation to change to the opposite and back again in

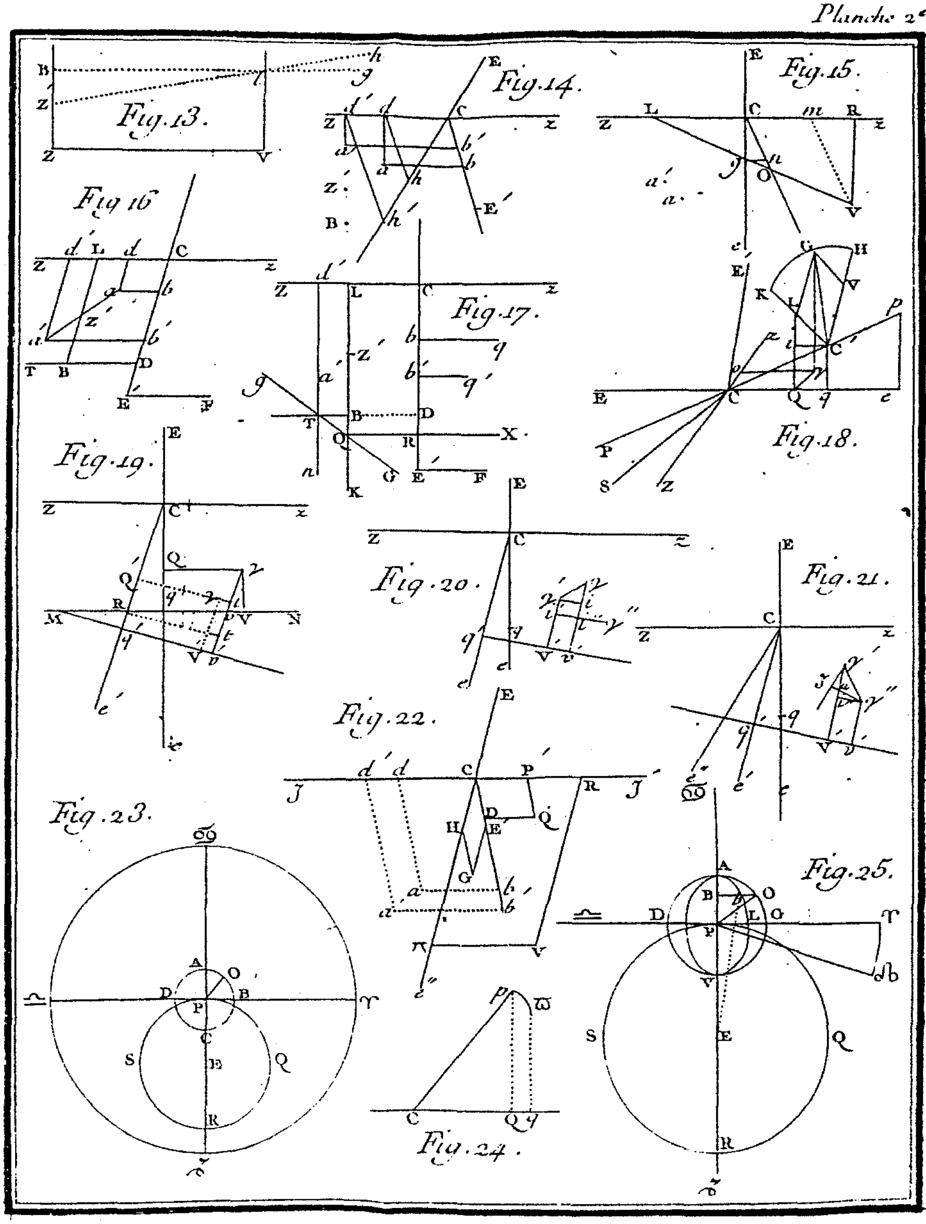


Fig. 8. A page with figures from d'Alembert's book (1749). His Fig. 25 shows the nutation ellipse, the major axis of which is APV, as well as the precession circle, with the diameter PER.

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lesquels P peut se développer, et par $\Sigma . k' . \sin.(it+\epsilon)$, la somme des termes dans lesquels P' peut se développer, Σ étant la caractéristique des intégrales finies; on aura

$$\frac{d\theta}{dt} = \left(\frac{A+B-2C}{2n.C}\right) . \Sigma . k' . \sin.(it+\epsilon); \qquad (H)$$

$$\frac{d\psi}{dt} . \sin.\theta = \left(\frac{2C-A-B}{2n.C}\right) . \Sigma . k . \cos.(it+\epsilon).$$

En intégrant ces équations, sans avoir égard aux constantes arbitraires; on aura les parties de θ et de ψ qui dépendent de l'action de l'astre L. Pour avoir les valeurs complètes de ces variables, il faut

Fig. 9. Laplace's precession-nutation formulae as they appear in the "Mécanique céleste" (1799). For the Earth $A = B$.

26 000 years, thus causing periodical climatic variations. At the same time the major axis of the Earth's orbit is moving, making the true period of these climatic variations 21 000 years. The present situation with longer winters in the southern hemisphere explains the large southern polar ice cap. Half the period ago, i.e. 10 500 years ago, the situation must have been the opposite – hence the Ice Age in the northern hemisphere at that time. Thus, according to Adhémar, the precession caused repeated glaciations, alternately in the northern and the southern hemispheres.

Adhémar's theory was modified in 1875 by *James Croll* (1821–1890), a Scottish geologist. Croll argued that the precession was insufficient to cause glaciations. However, the precession, in combination with the periodical variations of eccentricity and inclination of the Earth's orbit, could produce the glaciations, or at least trigger them. This hypothesis, which became more or less discarded towards the end of the century, has in our days received quite a lot of support (mainly through the theory of Milankovitch).

2.6. Precession-nutation and dynamical properties of the Earth

The first attempt to investigate the effects of the internal constitution of the Earth on precession and nutation was made as early as 1839. *William Hopkins* (1793–1866), an English mathematician, tried to compute these phenomena under the assumption of a fluid interior of the Earth. His conclusion was that the observed amount of precession required a thickness of the crust exceeding one fifth of the Earth's radius. Hopkins was a pioneer in applying mathematical methods to investigate a non-rigid Earth, complaining that he could not get geologists to understand mathematics nor mathematicians to take an interest in his geology.

Hopkins' result was supported by further arguments of his pupil *William Thom-*

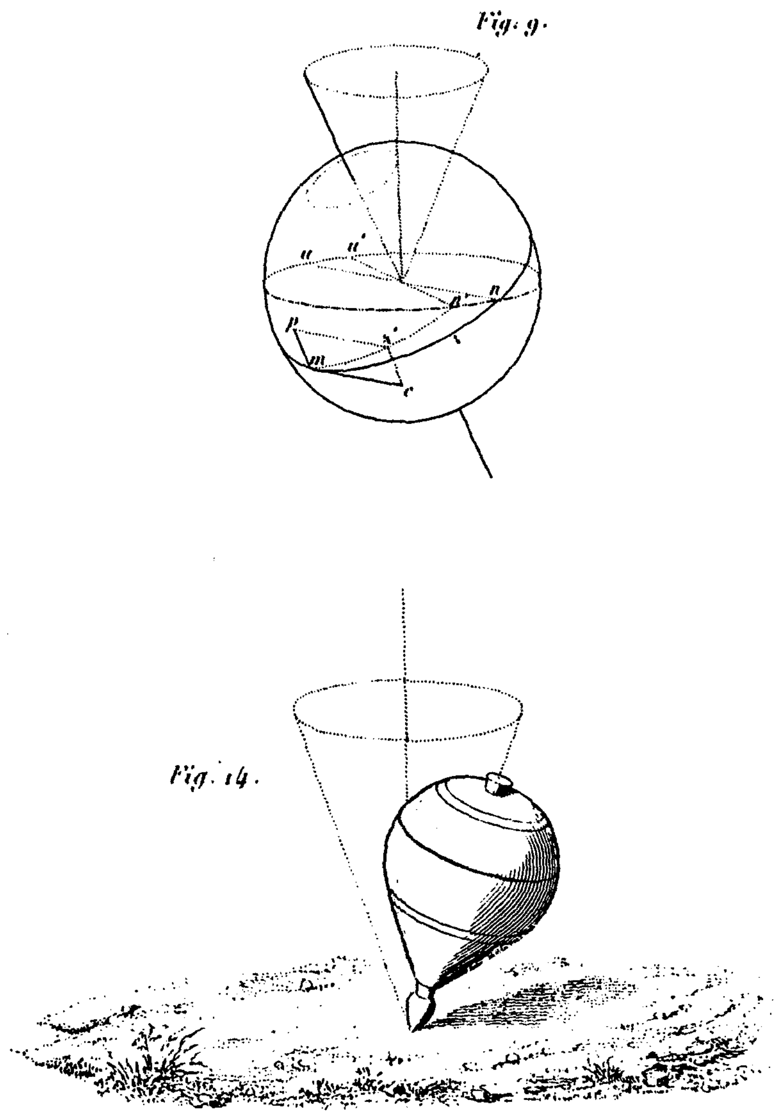

Fig. 10. The precession of the Earth compared to the precession of a spinning top. Illustrations used by Adhémar to explain his theory on precession and climate (1842).

son (1824–1907) – later *Lord Kelvin* – in the 1863 tidal paper treated in Section 1.5. Thomson devoted some paragraphs in this paper to precession and nutation, which he claimed would be drastically diminished if the Earth was not solid.

However, a visit to Simon Newcomb in America in 1876 (cf. Section 3.3) made him change his mind. On returning to England Thomson stated in a speech:

Admitting fully my evidence for the rigidity of the earth from the tides, he [Newcomb] doubted the argument from precession and nutation.... I could only stammer out that I had convinced myself that so-and-so and so-and-so, at which I had arrived by a non-mathematical short cut, were true . . . . But doubt entered my mind regarding the so-and-so and so-and-so; and I had not completed the night journey to Philadelphia which hurried me away from our unfinished discussion before I had convinced myself that they were grievously wrong. So now I must request as a favour that each one of you on going home will instantly turn up his or her copies of the Transactions of the Royal Society for 1863 . . . and draw the pen through §§ 21–32 of my paper on the 'Rigidity of the Earth'.

So far nothing can be considered as absolutely proved with reference to the interior solidity of the earth from precession and nutation.

Inspired by the discovery of polar motion several scientists tried to develop theories for a rotating ellipsoidal Earth containing liquid (see Section 3.3), but no very successful theory with respect to precession-nutation emerged until 1910. Then *Henri Poincaré* (1854–1912), the French mathematical physicist, investigated this complicated problem. Nine years earlier he had been able to modify Lagrange's equations so that they could be used not only for solids but also for liquids. Applying these equations Poincaré discovered that in a liquid core of about the same flattening as the Earth a resonance could occur which might perturb the nutation.

Since about the same time there was seismological evidence for liquid core surrounded by a mantle with a thin crust, the radius of the core being close to half the Earth's radius. Poincaré's theory combined with this knowledge formed the foundation upon which *Harold Jeffreys* (1891–1989), the English astronomer and geophysicist, built his first analyses of the dynamic effects of a liquid core. They were published in a series of papers starting in 1948. Jeffreys is here concerned with the significant discrepancy between the observed amplitude of the principal lunar nutation (9″.21) and the theoretical amplitude (9″.23), based on the assumption of a rigid Earth. He finds that the liquid core really will reduce the theoretical amplitude, but that it is reduced too much:

The main conclusion is that the theoretical nutation, obtained by taking the Earth as rigid, is probably too large; allowance for fluidity of the core while taking the shell as rigid reduces it to well below the observed value; but allowance for elasticity of the shell at the same time might result in agreement.

Jeffrey's suspicion that elasticity might bring the liquid core theory in better agreement with observations was confirmed by himself the following year (1949), but there was still not full agreement:

My previous conjecture that elasticity of the shell would reduce the effect of fluidity of the core is verified.

Even with these adjustments the effect remains too great, though the excess is possibly not greater than might be due to the simplifications made in the adopted model [of the Earth].

The work by Jeffreys formed a major step towards understanding precession-nutation of an elastic Earth with a liquid core. Moreover, it served as a starting point for a liquid core theory for earth tides (Section 1.6), and it will turn up again in connection with polar motion (Section 3.3).

48. En ſuppoſant F, G, H nulles, on a, comme on l'a vu dans l'article 42,

$$\frac{dT}{dp} = Ap, \quad \frac{dT}{dq} = Bq, \quad \frac{dT}{dr} = Cr,$$

& ces valeurs étant ſubſtituées dans les trois équations différentielles (A), il vient celles-ci,

$$dp + \frac{C-B}{A} qr\,dt = 0, \quad dq + \frac{A-C}{B} pr\,dt = 0, \quad dr + \frac{B-A}{C} pq\,dt = 0;$$

leſquelles s'accordent avec celles que M. Euler a employées dans la ſolution qu'il a donnée le premier de ce problême

Fig. 11. Euler's equations for a freely rotating body as presented by Lagrange in the "Mécanique analitique" (1788). For the Earth $A = B$.

3. History of Polar Motion Theory

3.1. The Mathematical Discovery of Polar Motion

The 18th century was a golden age for the application of mathematical methods to dynamical problems. *Leonhard Euler* (1707–1783), the Swiss mathematician working in Russia and Germany, was the one who derived equations for the rotation of rigid bodies. These equations are the "Euler equations" that we encountered in connection with precession-nutation in Section 2.4. They were announced in 1758 in one of Euler's many papers in the Transactions of the Royal Academy of Sciences in Berlin, "Du mouvement de rotation des corps solides autour d'un axe variable". By putting the right-hand sides of his equations equal to zero Euler obtains the equations for a freely rotating body, i.e. the case with no gravitational torques present. The equations then reveal that the axis of rotation is not fixed in the body – in other words, that the poles on the surface of the body, e.g; the Earth, must be moving.

A very elegant method for finding the equations of motion of, among other things, a rotating body was given by the Italian-French mathematician *Joseph Lagrange* (1736–1813) in his "Mécanique analitique", appearing in 1788. Euler's equations turned out to be a simple example of Lagrange's equations, which were based on the calculus of variations developed by Euler. It is in Lagrange's book we find Euler's equations for polar motion in the form that we use them today – see Figure 11. They show that the frequency, or period, is governed by the flattening of the Earth through the moment of inertia ratio $(C - A)/A$.

Neither Euler nor Lagrange seems to have discussed the resultant motion of the pole very much. Lagrange, whose successful ambition was to create an analytical dynamics free from geometrical methods, even declares in his preface: "One cannot find any figures in this work."

In contrast to this, *Louis Poinsot* (1777–1859), another French mathematician, concentrated on a purely geometrical method to study the motion of the rotational axis of a freely rotating body. His solution, published in two versions in 1834 and 1851, revealed that the axis of rotation could be looked upon as describing two cones closely associated with each other: One cone was formed by the motion of the rotational axis around the axis of maximum moment of inertia, or the symmetry axis of the body. The other cone was formed by the motion of the rotational axis around the axis of angular momentum fixed in space. Poinsot found that the body cone rolls without slipping around the space cone, the line of contact being the instantaneous axis of rotation. The body cone produces the Eulerian motion of the pole; the space cone represents a small free nutation.

3.2. The great surprise

During the second half of the 19th century an intense effort was made at several observatories to confirm the existence of a polar motion of the Earth. From Euler's equations it could be predicted that its period would be around 304 days; this had been done in 1844 by the German astronomer *Christian Peters* (1806–1880). The polar motion should manifest itself as a periodic variation of the latitude of the observatory.

Meanwhile *William Thomson* (1824–1907) – the English physicist we already know from earth tides and precession – pointed out, in 1876, that the motion of the pole might be more complicated than generally believed. Possible redistributions of matter in and on the Earth would influence the position of the pole, thereby preventing polar motion from damping. The seasonal redistribution of masses in the ocean and the atmosphere should cause a polar motion of its own. And the motion of the pole should raise a tide in the ocean. Altogether, Thomson expected the whole thing to be a quite irregular phenomenon.

Besides, the German geodesist *Friedrich Helmert* (1843–1917) claimed that a secular drift of the pole could occur. Its main cause, he argued in 1884, would be postglacial rebound.

The search for a variation of latitude went on for many years without success. When it finally was detected it was by a German astronomer, *Friedrich Küstner* (1856–1936), who was not looking for this effect at all. Küstner's purpose was to determine carefully the constant of aberration, by observing stars at the Berlin observatory. When his results turned out to be inconsistent he made a detailed investigation of them. It ended in the detection of a latitude variation amounting to a few tenths of a second of arc in about one year. His announcement of this in 1888 aroused great international interest.

Seth Carlo Chandler (1846–1913), a private astronomer in America, had been making stellar observations coincident in time with those of Küstner in Germany, the two of them, however, not knowing of one another's work. Chandler had observed the same effect as Küstner but, unlike him, had not dared to publish it. When reading Küstner's report Chandler immediately recognized the effect and started upon a thorough investigation not only of their own data but of all appropriate observation series made in the world during the last 50 years. In 1891, when he had analysed two long simultaneous series from America and Russia as well as the two series already mentioned, he was ready to publish a great surprise, in "On the Variation of Latitude" in the Astronomical Journal:

> Before entering upon the details of the investigations . . . it is convenient to say that the general result of a preliminary discussion is to show a revolution of the earth's pole in a period of 427 days, from west to east, with a radius of thirty feet, measured at the earth's surface.

The period was 427 days instead of 304! This was a most unexpected result and so contradictory to theory that many found it hard to believe.

A year later (1892) Chandler had completed investigations comprising material from no less than 17 observatories and was able to announce, in a later part of his above-mentioned paper:

> The observed variation of the latitude is the resultant curve arising from two periodic fluctuations superposed upon each other. The first of these, and in general the more considerable, has a period of about 427 days . . . . The second has an annual period.

Chandler's curve of the latitude variation is shown in Figure 12. The annual period was the one predicted by Thomson sixteen years earlier. But what was the 427-day-period? Chandler could offer no explanation.

Let us for a moment look back a little. During 50 years scientists had tried to find the polar motion – with no result. And then, suddenly, one man succeeds – using the same data which had led nowhere when in the hands of others. How could this come about? The answer is that people before Chandler were so convinced about the theoretical 304-day-period that they never looked for anything else. Chandler, on the other hand, had no respect for existing theories:

> I am not much dismayed by the argument of conflict with dynamic laws, since all that such a phrase means must refer merely to the existent state of the theory at any given time.

3.3. Polar motion and dynamical properties of the earth

Already the year after Chandler's discovery the American astronomer *Simon Newcomb* (1835–1909), in the Monthly Notices of the Royal Astronomical Society, presented an explanation of the surprisingly long period of polar motion. In an earlier paper Newcomb had admitted that the Chandler period was "in such disaccord with the received theory of the earth's rotation that, at first, I was disposed to doubt its possibility". But in his paper of 1892, "On the Dynamics of the Earth's Rotation, with respect to the Periodic Variations of Latitude" (see Figure 13), Newcomb writes:

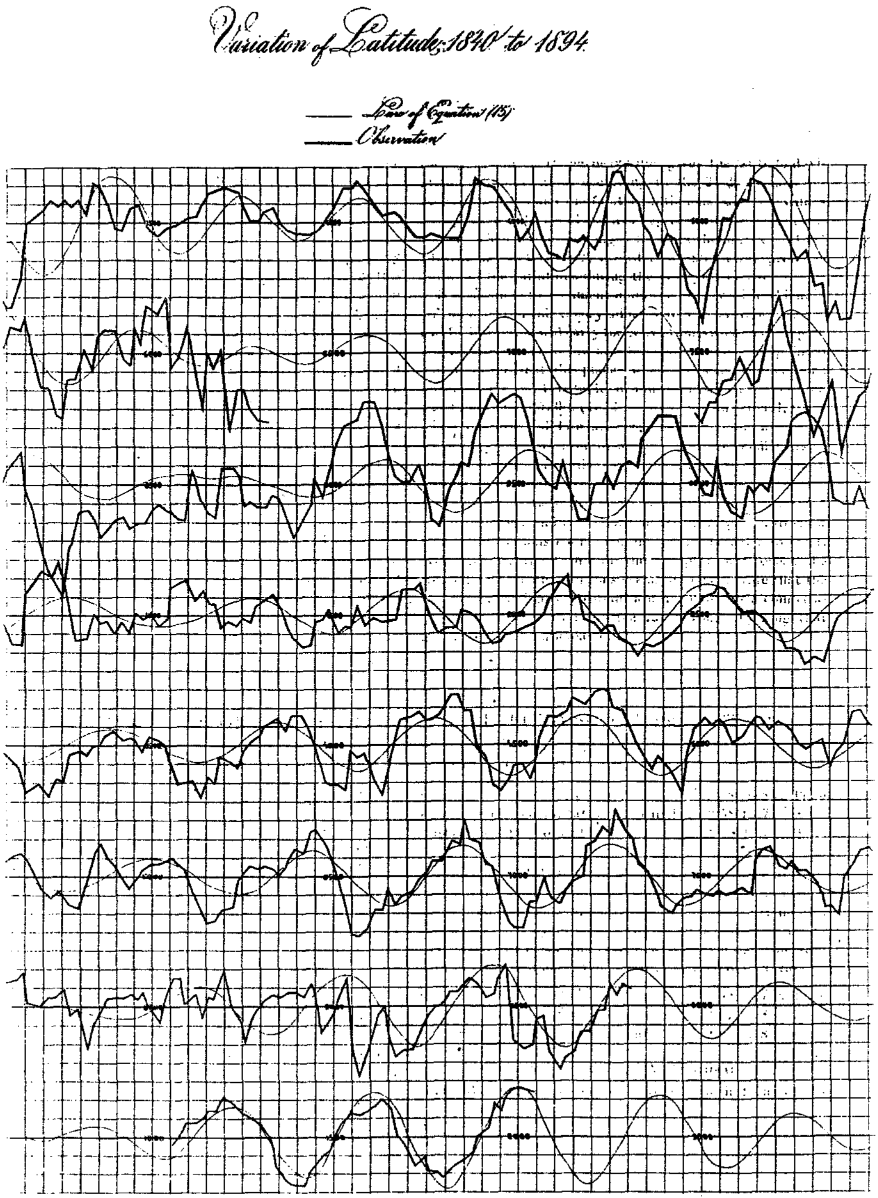

Fig. 12. The periodic latitude variations found by Chandler (1892).

Mr. Chandler's discovery gives rise to the question whether there can be any defect in the theory which assigns 306 days as the time of rotation. The object of this paper is to point out that there is such a defect – namely, the failure to take account of the elasticity of the Earth itself, and of the mobility of the ocean.

The main point here was the elasticity of the Earth. Only ten years earlier Darwin –

On the Dynamics of the Earth's Rotation, with respect to the Periodic Variations of Latitude. **By Simon Newcomb.**

The recent remarkable discovery of Mr. S. C. Chandler, that the axis of rotation of the Earth revolves around the axis of maximum moment of inertia in a period of about 427 days, is worthy of special attention.* At first sight it seems in complete contradiction to the principles of dynamics, which show that the ratio of the time of such a rotation to that of the Earth's revolution should be equal to the ratio of the polar moment of inertia of the Earth to the difference between the equatorial and the polar moments. Representing these moments by A and C, it is well known that the theory of rotation of a rigid body gives the equation

$$\tau = \frac{A}{C-A},$$

τ being the period of rotation of the pole in sidereal days.

Now the ratio in question is given with an error not exceeding a few hundredths of its total amount by the magnitude of the precession and nutation. The value found by Oppolzer is $\frac{1}{305}$, giving the time of rotation as 305 days.

This result has long been known, and several attempts have been made to determine the distance between the two axes, especially at Poulkova and Washington. A series of observations was made with the Washington Prime Vertical Transit during the years 1862–1867, including six complete periods of the inequality. Thus the determination of the coefficient and zero of the argument is completely independent of all sources of error having an annual or diurnal period. Such errors are

* *Astronomical Journal*, Numbers 248. 249.

Fig. 13. The beginning of Newcomb's paper announcing the explanation for Chandler's period of the polar motion (1892).

inspired by Thomson – had made the first numerical estimation of the Earth's elasticity, based on his discovery of the earth tides; see Section 1.5. Making use of this Newcomb showed, with a fairly simple line of reasoning, that the effect of elasticity is to lengthen the period of polar motion by about 100 days or somewhat more. This was in good accordance with the observations.

As we can see from the quotation above Newcomb also paid attention to the mobility of the ocean. He found that the ocean pole tide with an amplitude of the order of one cm will have the effect of lengthening the polar motion period by some 30 days. Putting the two effects together Newcomb arrived at a theoretical period of 443 days, only slightly exceeding Chandler's observed period of 427 days.

In 1893 *Francois Folie* (1833–1905), a Belgian astronomer, claimed that an increase of the period should be caused by a partial fluidity of the Earth's interior. A closer examination of this view was undertaken in 1895 by *Sydney Samuel*

Hough (1870–1923), an English astronomer working in South Africa for many years, and at the same time also by a Russian mathematician, *Fedor Sludsky* (1841–1897). Hough arrived at the unexpected result that a fluid core will shorten the period of polar motion --not lengthen it. Accepting Newcomb's explanation of the Chandler period he, therefore, concluded that a fluid core could not be very large. Thus Hough found that polar motion theory produced constraints on the dimensions of a fluid interior of the Earth, just as Hopkins earlier had suggested that precession theory did. It should also be noted that Hough, within this theory, was the first one to recognize the small free nutation of the core.

At the turn of the century elastic deformations of the Earth were known through two different phenomena: earth tides and polar motion. The tidal Love numbers h and k were treated in Section 1.6. Their numerical use was made possible by an important formula for polar motion. It was derived in 1909 by two British scientists, the physicist *Joseph Larmor* (1857–1942) and the geophysicist *Augustus Love* (1863–1940). Larmor and Love worked separately but simultaneously. Their two papers were read to the Royal Society of London on the same day. The formula they both had found relates the Chandler period T of the elastic Earth to the Euler period T_0 of a rigid Earth through the Love number k:

$$1 - \frac{T_0}{T} = k\frac{q}{2f - q}$$

(f being the flattening and q the equatorial ratio between centrifugal force and gravity). Their derivations, however, differed: While Love's presupposed a certain internal structure of the Earth, Larmor's was free from such a hypothesis and, hence, more generally valid. The formula yielded $k = 0.3$, a value which immediately could be used in tidal theory to calculate h.

Through the polar motion observations performed at the International Latitude Service a secular effect, as anticipated by Helmert, was revealed in 1922 by *Walter Lambert* (1879–1968), an American geodesist. The mean pole drifted slowly towards North America, at a rate of about half a second of arc per century. The possibility of this being a crustal movement in the opposite direction was ruled out by Lambert as being too large. The origin of the polar drift was unknown; it would take another half a century before it turned out that Helmert's postglacial rebound idea was, in fact, relevant.

The liquid core effect studied by Hough was taken up again in 1948 by *Harold Jeffreys* (1891–1989), the English astronomer and geophysicist we already know from his theories of tides and nutation. He now could give a numerical estimate: The liquid core of the size found by seismologists will shorten the polar motion period by about 30 days. This is about as much as the ocean will lengthen the period, as found by Newcomb, so that these two effects nearly cancel out.

With Jeffreys' liquid core paper, dealt with also in the context of nutation

(Section 2.6), and with Takeuchi's elastic tidal model (Section 1.6), the time had come for bringing the three related geodynamic phenomena of earth tides, precession-nutation and polar motion together into a common theory.

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References

A. Original works (in chronological order)

Strabon: 23 A.D. *Geographika*,Ameseia (English translation by H. L.Jones, Loeb Classical Library, 1917–1932).

Ptolemaios. K.: c. 150 A.D., *Mathematike syntaxis* ("Almagest"). Alexandria (English translation with annotations by G. J. Toomer, Duckworth & Co., 1984.)

Copernicus. N.: 1543, *De Revolutionibus Orbium Coelestium Libri VI*, Nürnberg. (English translation in Great Books of the Western World, 16, 1952.)

Galilei. G.: 1632, *Dialogo Sopra i Due Massimi Sistemi del Mondo*, *Tolemaico e Copernicano*, Firenze. (English translation, University of California Press, 1953.)

Wallis. J.: 1666, 'Hypothesis About the Flux and Reflux of the Sea', *Philosophical Transactions of the Royal Society* **1**, London.

Newton, I.: 1687, *Philosophiae Naturalis Principia Mathematica*, London.

Bernoulli, D.: 1740, Traité du Flux et Reflux de la Mer, (Referenced in d'Alembert, 1749; see below.)

Bradley, J.: 1748, 'A Letter to the Right honourable George Earl of Macclesfield Concerning an Apparent Motion observed in Some of the Fixed Stars', *Philosophical Transactions of the Royal Society* **45**, London.

d'Alembert, J.: 1749, *Recherches sur la Précession des Équinoxes, et sur la nutation de l'axe de la Terre, dans le Systeme Newtonien*, Paris.

Euler, L.: 1749, 'Recherches sur la Précession des Équinoxes, et sur la Nutation de l'axe de la Terre, *Histoire de l'Academie Royale des sciences et Belles Letters*, Berlin.

Kant, I.: 1754, 'Untersuchung der Frage: Ob die Erde in ihrer Umdrehung um die Achse, wodurch sie die Abwechselung des Tages und der Nacht hervorbringt, einige Veränderung seit den ersten Zeiten ihres Ursprunges erlitten habe, welches die Ursache davon sey, und woraus man sich ihrer versichern könne?', *Königsbergische wöchentliche Frag- und Anzeigungs-Nachrichten* **23–24**, Königsberg.

Euler, L.: 1758, 'Du Mouvement de Rotation des Corps Solides autour d'un Axe Variable', *Histoire de l'Académie Royale des Sciences et Belles Lettres*, Berlin.

Laplace, P. S.: 1775, 'Recherches sur Plusieurs Points du Système du Monde', *Histoire de l'Académie Royale des Sciences*, Paris.

Lagrange, J. L.: 1788, *Mécanique Analitique*, Paris.

Laplace, P. S.: 1799, *Traité de Mécanique Céleste*, *II*. Paris.

Poinsot, L.: 1834, 'Théorie Nouvelle de la Rotation des Corps', *L'Institut*, *Journal Général des Sociétés et Travaux Scientifiques* **2**, Paris.

Hopkins, W.: 1839, 1840, 1842, 'Researches in Physical Geology', *Philosophical Transactions of the Royal Society* **129**, **130**, **132**, London.

Adhémar, J.: 1842, *Révolutions de la Mer – Déluges Périodiques*, Paris. (2nd edition, Paris 1860.)

Peters, C. A. F.: 1844, 'Resultate aus Beobachtungen des Polarsterns am Ertelschen Verticalkreise der Pulkowaer Sternwarte, *Astronomische Nachrichten* **22**, Altona.
Poinsot, L.: 1851, 'Théorie Nouvelle de la Rotation des Corps', *Journal de Mathématiques Pures et Appliquées* **16**, Paris.
Ferrel, W.: 1853, 'On the Effect of the Sun and Moon upon the Rotatory Motion of the Earth', *Astronomical Journal* **3**, Cambridge.
Thomson, W.: 1863, 'On the Rigidity of the Earth', *Philosophical Transactions of the Royal Society* **153**, London.
Ferrel, W.: 1864, Note on the Influence of the Tides in Causing an Apparent Secular Acceleration of the Moon's Mean Motion', *Proceedings of the American Academy of Arts and Sciences* **6**, Washington.
Delaunay, C.: 1865, Sur l'Existence d'une Cause Nouvelle Ayant une Influence Sensible sur la Valeur de l'Équation Séculaire de la Lune', *Comptes Rendus Hebdomadaires de l'Académie des Sciences* **61**, Paris.
Airy, G. B.: 1866, 'On the Supposed Possible Effect of Friction in the Tides, in Influencing the Apparent Acceleration of the Moon's Mean Motion in Longitude', *Monthly Notices of the Royal Astronomical Society* **26**, London.
Thomson, W.: 1868, 'Committee for the Purpose of Promoting the Extension, Improvement and Harmonic Analysis of Tidal Observations', Report of the British Association for the Advancement of Science.
Croll, J.: 1875, *Climate and Time in their Geological Relations – A Theory of Secular Changes of the Earth's Climate*, London.
Thomson, W.: 1876, 'Opening Address to Section A of the British Association for the Advancement of Science', Report of the British Association for the Advancement of Science.
Darwin, G. H.: 1879, 1882, 'On the Precession of a Viscous Spheroid, and on the Remote History of the Earth', *Philosophical Transactions of the Royal Society* **170**, London.
Darwin, G. H.: 1882, 'A Numerical Estimate of the Rigidity of the Earth', *Nature* **27**. (Extended version in Section 848 in Thomson & Tait: *Treatise on Natural Philosophy*, 2nd edition, Oxford 1883.)
Darwin, G. H.: 1882, 'On Variations in the Vertical due to Elasticity of the Earth's Surface', *Philosophical Magazine and Journal of Science* **5/14**, London.
Helmert, F. R.: 1884, *Die mathematischen und physikalischen Theorieen der höheren Geodäsie, II.* Leipzig.
Küstner, F.: 1888, 'Neue Methode zur Bestimmung der Aberrations-Constante nebst Untersuchungen über die Veränderlichkeit der Polhöhe', *Beobachtungs-Ergebnisse der Königlichen Sternwarte* **3**, Berlin.
Chandler, S. C.: 1891, 1892, 'On the Variation of Latitude', *Astronomical Journal* **11**, **12**, Boston.
Newcomb, S.: 1891, 'On the Periodic Variation of Latitude, and the Observations with the Washington Prime-Vertical Transit', *Astronomical Journal* **11**, Boston.
Newcomb. S.: 1892, 'On the Dynamics of the Earth's Rotation, with Respect to the Periodic Variations of Latitude', *Monthly Notices of the Royal Astronomical Society* **52**, London.
Folie, F.: 1893, 'Expression Complète et Signification Véritable de la Nutation Initiale', *Acta Mathematica* **16**, Stockholm.
Hough, S. S.: 1895, 'The Oscillations of a Rotating Ellipsoidal Shell containing Fluid', *Philosophical Transactions of the Royal Society* **A 186**, London.
Sludsky, F.: 1895, 'De la Rotation de la Terre Supposée Fluide à son Intérieur', *Bulletin de Société Imperiale des Naturalistes* **9**, Moscow.
Harris, R. A.: 1904, 'Manual of Tides, IVB', Report of the United States Coast and Geodetic Survey, Washington.
Herglotz, G.: 1905, 'Ueber die Elastizität der Erde bei Berücksichtigung ihrer variablen Dichte', *Zeitschrift für Mathematik und Physik* **52**, Berlin.
Hecker, O.: 1907, 'Beobachtungen an Horizontalpendeln über die Deformation des Erdkörpers unter dem Einflusz von Sonne und Mond.' *Veröffentlichung des Königl. Preuszischen Geodätischen Institutes* **32**, Berlin.

Love, A. E. H.: 1909, 'The Yielding of the Earth to Disturbing Forces', *Proceedings of the Royal Society* **A82**, London.

Larmor, J.: 1909, 'The Relation of the Earth's Free Precessional Nutation to its Resistance against Tidal Deformation', *Proceedings of the Royal Society* **A82**, London.

Poincaré, H.: 1910, 'Sur la Précession des Corps Déformables', *Bulletin Astronomique* **27**, Paris.

Love, A. E. H.: 1911, *Some Problems of Geodynamics*, Cambridge.

Schweydar, W.: 1914, 'Beobachtung der Änderung der Intensität der Schwerkraft durch den Mond.' *Königl Preuszische Akademie der Wissenschaften, Physikalisch-mathematische Classe*, Berlin.

Taylor, G. I.: 1919, 'Tidal Friction in the Irish Sea', *Philosophical Transactions of the Royal Society* **A220**, London.

Jeffreys, H.: 1920, 'Tidal Friction in Shallow Seas', *Philosophical Transactions of the Royal Society* **A221**, London.

Hoskins, L. M.: 1920, 'The Strain of a Gravitating Sphere of Variable Densityand Elasticity', *Transactions of the American Mathematical Society* **21**, New York.

Doodson, A. T.: 1921, 'The Harmonic Development of the Tide-generating Potential', *Proceedings of the Royal Society* **A100**, London.

Lambert, W. D.: 1922, 'The Interpretation of Apparent Changes in Mean Latitude, at the International Latitude Stations', *Astronomical Journal* **34**, Albany.

Jeffreys, H.: 1928, 'Possible Tidal Effects on Accurate Time-keeping', *Monthly Notices of the Royal Astronomical Society, Geophysical Supplement* **2**, London.

Pekeris, C. L.: 1940, 'Note on Tides in Wells', *Travaux de l'Association Internationale de Géodésie* **16**, Paris.

Jeffreys, H.: 1948, 'The Earth's Core and the Lunar Nutation', *Monthly Notices of the Royal Astronomical Society* **108**, London.

Jeffreys, H: 1949, 'Dynamic Effects of a Liquid Core', *Monthly Notices of the Royal Astronomical Society* **109**, London.

Takeuchi, H.: 1950, 'On the Earth Tide of the Compressible Earth of Variable Density and Elasticity', *Transactions of the American Geophysical Union* **31**.

B. Works including history of tidal theory, precession-nutation theory, or polar motion theory

Berry, A.: 1898, *A Short History of Astronomy from Earliest Times through the Nineteenth Century*, London.

Birett, H: 1974, 'Zur Vorgeschichte der Newtonschen Theorie der Gezeiten', in Birett *et al.* (eds.), *Zur Geschichte der Geophysik*, Springer Verlag, Berlin.

Cartwright, D. E.: 1980, 'The Historical Development of Tidal Science, and the Liverpool Tidal Institute', in Sears and Merriman (eds.), *Oceanography the Past*, Springer Verlag, Berlin.

Darwin, G. H.: 1898, *The Tides and Kindred Phenomena in the Solar System*, London.

Deacon, M.: 1971, *Scientists and the Sea 1650–1900*, Academic Press, London.

Harris, R. A.: 1898, 'Manual of Tides, I', Report of the United States Coast and Geodetic Survey, Washington.

Lambeck, K.: 1980, *The Earth's Variable Rotation: Geophysical Causes and Consequences*, Cambridge University Press, Cambridge.

Lambert, W. D.: 1931, 'The Variation of Latitude', *Bulletin of the National Research Council* **78**, Washington.

Lisitzin, E.: 1974, *Sea Level Changes*, Elsevier Scientific Publishing Co., Amsterdam.

Marmer, H. A.: 1926, *The Tide*, New York.

Melchior, P.: 1978, *The Tides of the Planet Earth*, Pergamon Press, Oxford.

Munk, W. H. and MacDonald, G. J. F.: 1960, *The Rotation of the Earth: A Geophysical Discussion*, Cambridge University Press, Cambridge.

Neugebauer, O.: 1975, *A History of Ancient Mathematical Astronomy*, Springer Verlag, Berlin.

Rochester, M. G., Jensen, O. G., Smylie, D. E.: 1974, 'A Search for the Earth's 'Nearly Diurnal Free Wobble'', *Geophysical Journal of the Royal Astronomical Society* **38**, London.

Ronan, C.: 1983, *The Cambridge Illustrated History of the World's Science*, Cambridge University Press, Cambridge.

Todhunter, I.: 1873, *A History of the Mathematical Theories of Attraction and the Figure of the Earth*, London.

Vicente, R. O.: 1980, 'The Earth's Constitution and the Nutations', in Fedorov *et al.* (eds.), *Nutation and the Earth's Rotation*, D. Reidel Publishing Co., Dordrecht.

Wood, R. M.: 1985, *The Dark Side of the Earth: The Battle for the Earth Sciences, 1800–1980.* Allen & Unwin, London.